Homotopy Invariance

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1 Homology

1.1 Singular Homology

Definition 1.1. In n-dimensions, an **n-simplex** exists in $\mathbb{R}^m$ space, and consists of $n+1$ vertices $[v_0, \cdots, v_n]$ such that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. The standard n-simplex is represented:

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \Sigma_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

Definition 1.2. Springing from the definition of n-simplex the **singular n-simplex** is a map $\sigma : \Delta^n \longrightarrow X$. This map can allow for disfiguration of the vertices of the n-simplex and need only be continuous, that is to say the image of the singular n-simplex map need not look like a simplex .

Definition 1.3. The **singular n-chain** is a formal sum $\sum\limits_i n_i\sigma_i$, where the σ_i are singular n-simplices, with coefficients $n_i \in \mathbb{Z}$, however it is not addition in the traditional summation. This formal sum is a finite collection of singular n-simplices in X with integer multiplicities.. The singular n-chains are elements of $C_n(X)$. This is the free abelian group comprised of all of such singular n-chains spanned by the basis set of singular n-simplices.

Definition 1.4. We define the **boundary homomorphism** for a singular n-simplex on X as: $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$, defined by:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

Since we define this function for a singular n-simplex, it extends linearly to n-chains, since n-chains are linear sums of singular n-simplices.

Lemma 1.5. *The composition* $C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$ *is zero.*

Proof. Given $\partial_n(\sigma) = \sum\limits_i (-1)^i \sigma \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$ then
$\partial_{n-1}\partial_n(\sigma) = \sum\limits_{j<i} (-1)^i(-1)^j \sigma[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] +$
$\sum\limits_{j>i} (-1)^i(-1)^{j-1} \sigma[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] = 0$
the two summations cancel because when you switch the i and j the second summation becomes the negative of the first summation. ∎

Definition 1.6. What we can now generalize from this is a sequence of homomorphisms between abelian groups, where the C_n groups have basis elements that are singular n-simplices, that together form the singular n-chains:

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial_{n+1}} C_{n-1}(X) \xrightarrow{\partial_n} \cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

where $\partial_n \partial_{n+1} = 0$, $\forall n$. This sequence is called a **chain complex**.

Definition 1.7. The property that the composition of two adjacent homomorphisms equaling zero is the same as the property that $Im\partial_{n+1} \subset Ker\partial_n$. An n-**cycle** is an element of $Ker\partial_n$. An n-**boundary** is an element of $Im\partial_{n+1}$. Using "mod" we create **homology classes** by $\operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$, which are the cosets of $\operatorname{Im}\partial_{n+1}$. If two cycles in $\operatorname{Im}\partial_{n+1}$ represent the same homology class (coset) they are said to be **homologous**. This also implies that the two cycles differ by a boundary. Thus we define the n^{th} **singular homology group** as $H_n(X) = Ker\partial_n / Im\partial_{n+1}$.

Proposition 1.8. *Corresponding to the decomposition of a space X into its path components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\oplus_\alpha H_n(X_\alpha)$.*

Proof. When we decompose X into X_α, the singular simplex maintains the path-connectedness therefore, $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. The boundary maps between the $C_n(X)$ also preserve the direct sum decomposition $C_n(X_\alpha) \xrightarrow{\partial_n} C_{n-1}(X_\alpha)$. Then the $\operatorname{Ker}\partial_n$ and the $\operatorname{Im}\partial_{n+1}$ split accordingly, therefore the homology groups split and are maintained, $H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha)$. ∎

Proposition 1.9. *If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. Hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. Recall that $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$ since $\partial_0 = 0$. Define $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ by $\sum\limits_i (n_i\sigma_i) = \sum\limits_i n_i$ the homomorphism taking each n-chain in C_0 to an integer.Since we assumed X to be nonempty, this is surjective, thus it is enough to show that $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$.
Notice that $\sigma : \Delta^1 \longrightarrow X$ the map from of a singular 1-simplex provides the following: $\epsilon\partial_1(\sigma) = \epsilon(\sigma \mid [v_1] - \sigma \mid [v_0]) = 1 - 1 = 0$ thus $\operatorname{Im}\partial_1 \subset \operatorname{Ker}\epsilon$
Suppose $\epsilon(\sum\limits_i n_i\sigma_i) = 0$ then $\sum\limits_i n_i = 0$. In this case the σ_i's are singular 0-simplices, or rather single vertices in X. Choose $\tau_i : I \longrightarrow X$ from a start vertice x_0 the image of σ_0, to $\sigma_i(v_0)$. Thinking of τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \longrightarrow X$, then it is clear that $\partial\tau_i = \sigma_i - \sigma_0$. Therefore since we assumed $\sum\limits_i n_i = 0$ then $\partial(\sum\limits_i n_i\tau_i) = \sum\limits_i n_i\sigma_i - \sum\limits_i n_i\sigma_0 = \sum\limits_i n_i\sigma_i$ so it is a boundary and $\operatorname{Ker}\epsilon \subset \operatorname{Im}\partial_1$.
Hence $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$ and $H_0(X) \approx \mathbb{Z}$. ∎

Proposition 1.10. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof. Since X is a point, there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum\limits_i (-1)^i \sigma_{n-1}$, is the sum of $n+1$ terms. This sum is 0 for n odd, and σ_{n-1} for n even and non-zero. This creates the chain complex with alternating boundary maps of isomorphisms and the trivial map:

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

The homology groups of this chain complex are all trivial except for $H_0 \approx \mathbb{Z}$. ∎

Remark. For the next section we will talk about functions between spaces, and the induced homomorphism between the homology groups of the spaces:

$$f : X \longrightarrow Y \text{ induces } f_* : H_n(X) \longrightarrow H_n(Y)$$

This takes some showing in the intermediate step between the C_n chains and boundary maps. which creates a lattice like diagram.

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"'] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"'] & \dots \\
\dots \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \arrow[r, "\partial"'] & \dots
\end{tikzcd}$$

Definition 1.11. Within these lattice structures we can see the **commutative diagram** that have the property that composing two maps together,and then 'reversing' the composition yields an equivalent result and is denoted $f_\sharp\partial = \partial f_\sharp$. For example using the lattice diagram to illustrate, if we start at $C_n(X)$ go down using $f_\sharp$ to $C_n(Y)$, then go right using ∂ to $C_{n-1}(Y)$, we have just performed $f_\sharp\partial$. So it is equivalent to start at $C_n(X)$ go right to $C_{n-1}(X)$ using ∂, and then down to $C_{n-1}(Y)$ using $f_\sharp$, thus performing $\partial f_\sharp$. Beginning and ending in the same groups, but having a slightly different middle route.

Definition 1.12. Given a map $f : X \longrightarrow Y$ the induced homomorphism $f_\sharp : C_n(X) \longrightarrow C_n(Y)$ is defined by composing each singular n-simplex with f such that $f_\sharp(\sigma) = f\sigma : \Delta^n \longrightarrow Y$. This induced homomorphism is shown on the commutative diagram, and the property that $f_\sharp\partial = \partial f_\sharp$ is also defined by saying that the $f_\sharp$'s define a **chain map** from the chain complex of X to the chain complex of Y. A chain map $f_\sharp$ induces a map f^* on homology.

Theorem 1.13. *If two maps $f, g : X \longrightarrow Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \longrightarrow H_n(Y)$*

Proof. If we think about the inclusion maps $i_j : X \longrightarrow X \times (0,1)$ sending $x \longrightarrow (x, j)$ where $j = \{0,1\}$ these maps are homotopic. So if we could show $(i_0)_* = (i_1)_*$, with homotopy H between f and g then it is proved, since:

$$(f)_* = (H \circ i_0)_* = H_* \circ (i_0)_* = H_* \circ (i_1)_* = (H \circ i_1)_* = (g)_*$$

then it is left to show that $(i_0)_* = (i_1)_*$.
To prove this it is enough to show that $\exists h : Z_n(X) \longrightarrow C_{n+1}(X \times (0,1))$ such that $\partial h(c) = (i_1)_\sharp c - (i_0)_\sharp c \ \forall n$.
In fact we will construct such an $h : C_n(X) \longrightarrow C_{n+1}(X \times (0,1))$ such that $h\circ\partial + \partial\circ h = (i_1)_\sharp - (i_0)_\sharp$.
Such a map h is called a homotopy operator.
Construct h:
Examining the cylinder $\Delta_n \times [0,1]$ and its vertices denoted $\Delta_n \times \{0\}$ by $E_i = (e_i, 0)$ and $\Delta_n \times$

$\{1\}$ by $E_i' = (e_i, 1)$

Also denote the singular $(n+1)$-simplex given by the complex hull of $\{E_0, E_1, \cdots, E_i, E_i', \cdots, E_n'\}$ as:

$$G_{i,n}(e_j) = \begin{cases} E_j & \forall j \leq i \\ E_{j-1}' & \forall j > i \end{cases}$$

Then we can define $h(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \times \mathbb{1}) \circ G_{i,n} \ \forall n$

For the next two claims we will be composing $G_{i,n} : \Delta_{n+1} \longrightarrow \Delta_n \times \mathbb{1}$ with $F_{i,n} : \Delta_{n-1} \longrightarrow \Delta_n$ where F is defined as the face map such that

$$F_{i,n}(e_j) = \begin{cases} e_j & \forall j < i \\ e_{j+1} & \forall j \geq i \end{cases}$$

Claim 1: To show that $G_{j,n} \circ F_{j,n+1} = G_{j-1,n} \circ F_{j,n+1}$

For me it is easiest to see that this is true if we break down what each function does individually and then compose them. So we have, in essence three functions:

$$F_{j,n+1}(e_k) = \begin{cases} e_k & 0 \leq k < j \\ e_{k+1} & j \leq k \leq n \end{cases}$$

$$G_{j,n}(e_k) = \begin{cases} E_k & 0 \leq k \leq j \\ E_{k-1}' & j < k \leq n+1 \end{cases}$$

$$G_{j-1,n}(e_k) = \begin{cases} E_k & 0 \leq k \leq j-1 \\ E_{k-1}' & j-1 < k \leq n+1 \end{cases}$$

Thus when we compose the left hand side:

$$G_{j,n}(F_{j,n+1}(e_k)) = \begin{cases} E_k & 0 \leq k < j \\ E_k' & j \leq k \leq n \end{cases}$$

And composing the right hand side:

$$G_{j-1,n}(F_{j,n+1}(e_k)) = \begin{cases} E_k & 0 \leq k < j \\ E_k' & j \leq k \leq n \end{cases}$$

We can see that the left hand side and right hand side agree, therefore $G_{j,n} \circ F_{j,n+1} = G_{j-1,n} \circ F_{j,n+1}$ and we have that claim 1 is true.

Claim 2: To show that

$$(F_{j,n} \times \mathbb{1}) \circ G_{i,n-1} = \begin{cases} G_{i+1,n} \circ F_{j,n+1} & \text{if } i \geq j \\ G_{j,n} \circ F_{j+1,n+1} & \text{if } i < j \end{cases}$$

Again for me it easiest to see what is happening when we look at each function individually. In this case there are six functions. Note that for $(F_{j,n} \times \mathbb{1})$ the function takes things of the form $(e_k, 0) = E_k$ or $(e_k, 1) = E'_k$, notationally I write this as d where $d \in (0, 1)$. The functions we are concerned with on the left hand side are

$$(F_{j,n} \times \mathbb{1})(e_k, d) = \begin{cases} (e_k, d) & 0 \leq k < j \\ (e_{k+1}, d) & j \leq k \leq n-1 \end{cases}$$

$$G_{i,n-1}(e_k) = \begin{cases} E_k & 0 \leq k \leq i \\ E'_{k-1} & i < k \leq n \end{cases}$$

Composing these two we see that there are two cases, with three subcases each. The two subcases are $i \geq j$ and $i < j$

$$\text{for } i \geq j : (F_{j,n} \times \mathbb{1})(G_{i,n-1}(e_k)) = \begin{cases} E_k & 0 \leq k < j \\ E_{k+1} & j \leq k \leq i \\ E'_k & i < k \leq n \end{cases}$$

and

$$\text{for } i < j : (F_{j,n} \times \mathbb{1})(G_{i,n-1}(e_k)) = \begin{cases} E_k & 0 \leq k \leq i \\ E'_{k-1} & i < k \leq j \\ E'_k & j < k \leq n \end{cases}$$

So it remains to show that the right hand side of the claim agrees with the cases of the left hand side. So for $i \geq j$ we have

$$G_{i+1,n}(e_k) = \begin{cases} E_k & 0 \leq k \leq i+1 \\ E'_{k-1} & i+1 < k \leq n+1 \end{cases}$$

$$F_{j,n+1}(e_k) = \begin{cases} e_k & 0 \leq k < j \\ e_{k+1} & j \leq k \leq n \end{cases}$$

So composing these two functions we get

$$G_{i+1,n}(F_{j,n+1}(e_k)) = \begin{cases} E_k & 0 \le k < j \\ E_{k+1} & j \le k \le i \\ E'_k & i < k \le n \end{cases}$$

Which agrees with the first case of the left hand side where $i \ge j$. Then for $i < j$ we have

$$G_{i,n}(e_k) = \begin{cases} E_k & 0 \le k \le i \\ E'_{k-1} & i < k \le n+1 \end{cases}$$

$$F_{j+1,n+1}(e_k) = \begin{cases} e_k & 0 \le k < j+1 \\ e_{k+1} & j+1 \le k \le n \end{cases}$$

So composing these two functions we get

$$G_{i,n}(F_{j+1,n+1}(e_k)) = \begin{cases} E_k & 0 \le k \le i \\ E'_{k-1} & i < k \le j \\ E'_k & j < k \le n \end{cases}$$

Which agrees with the second case of the left hand side where $i < j$. Thus claim 2 is true. So all that is left to show is that h is a homotopy operator.

Fix σ a singular complex of X

So

$$
\begin{aligned}
h(\partial\sigma) &= h(\sum_{j=0}^{n}(-1)^j\sigma\circ F_{j,n}) && (1.1)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{j+i}((\sigma\circ F_{j,n})\times \mathbb{1})\circ G_{i,n-1} && (1.2)\\
&= \sum_{j=0}^{n}\sum_{i=j}^{n-1}(-1)^{j+i}((\sigma)\times \mathbb{1})\circ G_{i+1,n}\circ F_{j,n+1} && (1.3)\\
&+ \sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^{j+i}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{j+1,n+1} \text{ by claim 2} && (1.4)\\
\partial(h(\sigma)) &= \partial(\sum_{i=0}^{n}(-1)^i(\sigma\times \mathbb{1})\circ G_{i,n} && (1.5)\\
&= \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{j+i}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{j,n+1} && (1.6)\\
&= \sum_{j=0}^{n}\sum_{i=j}^{n}(-1)^{j+i}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{j,n+1} && (1.7)\\
&+ \sum_{j=}^{n}\sum_{i=1}^{j-1}(-1)^{j+i}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{j,n+1} && (1.8)\\
&+ \sum_{i=0}^{n}(-1)^{i+n+1}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{n+1,n+1} && (1.9)\\
&= \sum_{j=0}^{n}\sum_{i=j}^{n-1}(-1)^{j+i+1}((\sigma)\times \mathbb{1})\circ G_{i+1,n}\circ F_{j,n+1} + \sum_{j=0}^{n}((\sigma)\times \mathbb{1})\circ G_{j,n}\circ F_{j,n+1} && (1.10)\\
&+ \sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^{j+i+1}((\sigma)\times \mathbb{1})\circ G_{i,n}\circ F_{j+1,n+1} + \sum_{j=0}^{n}(-1)^{2j+1}((\sigma)\times \mathbb{1})\circ G_{j,n}\circ F_{j+1,n+1} && (1.11)\\
& && (1.12)
\end{aligned}
$$

Note that there is a change of indices in line 1.7 ($i \to i+1$) and in line 1.8 ($j \to j+1$)

Then when we put this all together $h(\partial\sigma) + \partial(h(\sigma))$ we get a lot of canceling. Line 1.3 cancels with the first part of line 1.10, and line 1.4 cancels with the first part of 1.11. So we are left with:

$$h(\partial\sigma) + \partial(h(\sigma)) = \sum_{j=0}^{n}((\sigma)\times \mathbb{1})\circ G_{j,n}\circ F_{j,n+1} - \sum_{j=0}^{n}((\sigma)\times \mathbb{1})\circ G_{j,n}\circ F_{j+1,n+1}$$

So with one more change of indices ($j \to j-1$) for the second summation

$$h(\partial\sigma) + \partial(h(\sigma)) = \sum_{j=0}^{n} ((\sigma) \times \mathbb{1}) \circ G_{j,n} \circ F_{j,n+1} - \sum_{j=1}^{n+1} ((\sigma) \times \mathbb{1}) \circ G_{j-1,n} \circ F_{j,n+1}$$

Then by claim 1:

$$= ((\sigma) \times \mathbb{1}) \circ G_{0,n} \circ F_{0,n+1} - ((\sigma) \times \mathbb{1}) \circ G_{n,n} \circ F_{n,n+1} \tag{1.13}$$

$$= (i_1)_\sharp(\sigma) - (i_0)_\sharp(\sigma) \tag{1.14}$$

■

Corollary 1.14. *The maps $f_* : H_n(X) \longrightarrow H_n(Y)$ induced by a homotopy equivalence $f : X \longrightarrow Y$ are isomorphisms for all n.*